Energy level statistics in the transition regime between integrability and chaos for systems without an anti-unitary symmetry

1999 J. Phys. A: Math. Gen. 32 1427

(http://iopscience.iop.org/0305-4470/32/8/011)

J. Phys. A: Math. Gen. **32** (1999) 1427–1438. Printed in the UK PII: S0305-4470(99)97646-1

Energy level statistics in the transition regime between integrability and chaos for systems without an anti-unitary symmetry

Marko Robnik†‖, Jure Dobnikar‡¶ and Tomaž Prosen§[+]

† Center for Applied Mathematics and Theoretical Physics, University of Maribor, SI-2000 Maribor, Slovenia

‡ Jozef Stefan Institute, SI-1001 Ljubljana, Slovenia

§ Faculty of Mathematics and Physics, University of Ljubljana, SI-1000 Ljubljana, Slovenia

Received 16 September 1998

Abstract. Energy spectra of a particle with mass m and charge e in the cubic Aharonov–Bohm billiard containing around 10^4 consecutive levels starting from the ground state have been analysed. The cubic Aharonov–Bohm billiard is a plane billiard defined by the cubic conformal mapping of the unit disc pervaded by a point magnetic flux through the origin perpendicular to the plane of the billiard. The magnetic flux does not influence the classical dynamics, but breaks the anti-unitary symmetry in the system, which affects the statistics of energy levels. By varying the shape parameter λ the classical dynamics goes from integrable ($\lambda = 0$) to fully chaotic ($\lambda = 0.2$; Africa billiard). The level spacing distribution $P(S)$ and the number variance $\Sigma^2(L)$ have been studied for 13 different shape parameters on the interval ($0 \leqslant \lambda \leqslant 0.2$). Gaussian unitary ensembles statistics has proven correct for the completely chaotic case, while in the mixed regime the fractional power-law level repulsion has been observed. The exponent of the level repulsion has been analysed and is found to change smoothly from 0 to 2 as the dynamics goes from integrable to ergodic. This is precisely the analogy of the fractional power-law level repulsion observed in the Poisson–Gaussian orthogonal ensemble (GOE) transition by Prosen and Robnik (1993, 1994) and it thus essentially differs from the prediction of the random matrix theories. The semiclassical Berry–Robnik theory is expected to be correct in the ultimate semiclassical limit. However, we argue that the semiclassical regime has not been reached and give an estimate for the number of energy levels required for the Berry–Robnik statistics to apply.

The objective of this paper is to study the energy level statistics of a system without anti-unitary symmetry, in the transition region between classical integrability and full chaos (ergodicity). This is an important issue and deserves a few introductory explanations.

As for the spectral statistics in such a regime we expect some kind of a transition between Poisson and Gaussian unitary ensembles (GUE) statistics. Like in the case of Poisson–GOE, one might try to model this transition by the appropriate random matrix ensembles, as has been performed by Tomsovic [1], French *et al* [2], Leyvraz and Seligman [3] and most recently by Guhr and Müller-Groeling [4], where one studies the statistics of a sum of a Poisson diagonal matrix plus a GUE matrix. In [4] novel supersymmetric methods are used to solve this problem with new analytic results. At sufficiently small spacings this model leads unavoidably to

‖ E-mail address: robnik@uni-mb.si

¶ E-mail address: jure.dobnikar@ijs.si

[+] E-mail address: prosen@fiz.uni-lj.si

0305-4470/99/081427+12$19.50

quadratic level repulsion, in precise analogy to the linear level repulsion in the Poisson–GOE case as shown analytically by Robnik [5] for two-dimensional matrices, by Izrailev [6] for four-dimensional matrices and numerically by Prosen [7] for high-dimensional matrices. The main point of this paper is to demonstrate that the above modelling and its conclusions are generally correct but only on an exponentially small (energy level spacing) scale $\Delta S \propto \exp(-\text{const}/\hbar)$, as has been recently explained by Robnik and Prosen [8]. At energy ranges larger than ΔS, we in fact observe, the fractional power-law level repulsion where $P(S) \propto S^\beta$ at sufficiently small but not too small S, namely $\Delta S \leqslant S \leqslant 1$, where β can be different from 0 (Poisson), 1 (GOE) or 2 (GUE), and thus can take on any values in the interval $[0, 2]$. This has been firmly demonstrated for the Poisson–GOE case by Prosen and Robnik [9, 10], and is the main subject of this work where we have the Poisson–GUE transition in a dynamical system at not too high energies. The origin of the fractional power-law level repulsion is identified with the phenomenon of the (dynamical) localization of chaotic (irregular) quantal states and with the quantal Kolmogorov–Arnold–Moser (KAM) structure as observed in [9, 10]. Therefore this work and our numerical results are by no means a mere repetition or reformulation of previous mathematical models of the random matrix theories, but describe the aspects of dynamical localization effects. For a recent review see [25].

Introducing a magnetic field to a plane billiard breaks the time reversal symmetry. However, if there are space symmetries present, there can be a combined space–time symmetry in the system, which is anti-unitary [11]. Therefore, a billiard with no space symmetries is required. The natural choice is the cubic billiard defined by the conformal mapping of the unit disc ($|z| \leqslant 1$)

$$w = z + \lambda z^2 + \lambda e^{i\frac{\pi}{3}} z^3 \tag{1}$$

onto the physical plane w, with the shape parameter λ. A point magnetic flux through the origin perpendicular to the plane of the billiard is introduced. Such billiard systems are called Aharonov–Bohm billiards. The classical dynamics is unaffected by the point flux† and the system has the scaling property, which says that the classical dynamics is the same at all energies. The particle is therefore considered to have a unit speed. This family of billiards was introduced and first studied by Berry and Robnik [12].

Energy level statistics in the limiting, fully chaotic case of an Africa billiard ($\lambda = 0.2$), has been found to be consistent with the GUE of random matrices [13, 14]. We should stress that there is no rigorous proof that the classical motion in an Africa billiard is really ergodic. However, even a very careful numerical study cannot detect any significant island of stable/regular motion in the classical phase space. In this paper we have investigated the energy level statistics in the regime of mixed classical dynamics and have found *the fractional power-law level repulsion*‡, while we argue that the semiclassical theory of Berry and Robnik [15] applies for much larger sequential quantum numbers (see also [8]), in our case say 10^8. We also reconfirm [12, 13] the quadratic level repulsion and the validity of GUE (level spacing distribution and number variance) statistics in the ergodic case with high statistical accuracy.

The Schrödinger equation for the system [12] is

$$\begin{aligned} &\nabla^2\psi(r,\phi) - \frac{2i\alpha}{r^2}\frac{\partial\psi}{\partial\phi} - \frac{\alpha^2}{r^2}\psi(r,\phi) + k^2\left|\frac{dw}{dz}\right|^2\psi(r,\phi) = 0 \\ &\psi(1,\phi) = 0. \end{aligned} \tag{2}$$

† Only the trajectories which pass through the origin exactly are affected, but they have measure zero among the trajectories.

‡ The phenomenon of fractional power-law level repulsion has been typically found in mixed systems with anti-unitary symmetry and not too high energies (see [10, 9] and the references therein).

Here $k^2 = \frac{2mE}{\hbar}$ where E is the energy and the parameter $\alpha = \frac{e\Phi}{2\pi\hbar}$, where Φ is the magnitude of the magnetic flux and $2\pi\hbar$ is the Planck's constant. By (r, ϕ) we denote the polar coordinates in the z-plane. The relevant region of the values of α is $[0, \frac{1}{2}]$. It can easily be shown [11–13] that the anti-unitary symmetry is still present for integer and half integer values of α, so the most appropriate value is $\alpha = \frac{1}{4}$. We exclusively choose this value in our present calculations. In the integrable case ($\lambda = 0$) we expect to find Poissonian energy level distribution

$$P(S) = \mathrm{e}^{-S} \tag{3}$$

while in the completely chaotic case $P(S)$ is predicted by the GUE of the random matrix theory. The Wigner approximation† gives for GUE

$$P(S) = \frac{32}{\pi^2} S^2 \mathrm{e}^{-\frac{4}{\pi}S^2}. \tag{4}$$

For small S we have $P(S) \propto S^2$, i.e. quadratic level repulsion.

Let us first describe a semiclassical argument (see [16]) to estimate the number of converged energy levels in our calculation. By 'converged' we mean an accuracy of at least one per cent of the mean level spacing. We expressed the matrix elements of the Hamiltonian H in a basis defined by eigenvectors of the integrable Hamiltonian H^0 (circular billiard, $\lambda = 0$, see [12, 17]). We truncated the basis at the N^0th vector and diagonalized the finite ($N^0 \times N^0$) matrix. The eigenvectors $|n\rangle$ of H^0 are defined as $H^0|n\rangle = e_n|n\rangle$ and the eigenvalues of H as $H|\alpha\rangle = E_\alpha|\alpha\rangle$. The question is, how many accurate energy levels E_α associated with H are there in such a calculation? In the semiclassical limit the Wigner transform of an eigenstate with energy E is localized on the classical energy surface (see [18, 19] and the references therein).

This yields that an energy level E_α of H will converge if all eigenvectors $|n\rangle$ of H^0 with the energy surfaces $H^0(\vec{q}, \vec{p}) = E_n^0$ intersecting the energy surface of H, $H(\vec{q}, \vec{p}) = E_\alpha$, are present in the truncated basis‡. Geometrical consideration of foliation of energy surfaces (see [18] and the references therein) gives

$$\frac{N}{N^0} = r(\lambda) = \frac{A(\lambda)}{\pi \max_{|z|\leqslant 1} |\frac{\mathrm{d}w}{\mathrm{d}z}|^2} \tag{5}$$

where $A(\lambda)$ is the area enclosed by the billiard boundaries and is equal to $A(\lambda) = \pi(1+5\lambda^2)$ for the cubic billiard. The fraction of the converged levels as a function of λ is plotted in figure 1. Approximately one third of the levels are correct at $\lambda = 0.2$, whereas at $\lambda = 0.05$ there are about two thirds of good levels. As mentioned above, by a good level we mean an accuracy of at least one per cent of the mean level spacing. Of course, this *a priori* criterion is helpful, but the final judgement of the precision was based on the actual numerical convergence of the levels.

If the basis functions are arranged in a 'smart' way, the ($N^0 \times N^0$) matrix which has to be diagonalized has a band structure. In fact we do not diagonalize the Hamiltonian, but its inverse H^{-1} [20] which can have the desired band structure. The rearrangement of the basis is described in [9]. The CPU time demand of the diagonalization drops from $\mathcal{O}(N^3)$ to $\mathcal{O}(N^{2.5})$ due to the rearrangement. The size of the matrix was $N^0 = 18\,000$ for $\lambda = 0.2$ and $N^0 = 15\,500$ for 13 other shape parameters from table 1. By diagonalizing the band matrices the 14 spectra have been obtained containing $r(\lambda)N^0$ accurate levels.

† We tested the quality of the Wigner approximation by comparing it to the exact infinitely-dimensional solution (see [24]) and found that Wigner approximation is almost perfectly correct, the tiny deviations from the exact solution are two orders of magnitude smaller than the deviations of our numerical results.

‡ It can be shown that the terms in the Hamiltonian (2) containing α are of the second order in $\hbar$, so the magnetic flux does not influence the density of states in the semiclassical limit (of course, it mixes the levels and affects the statistics).

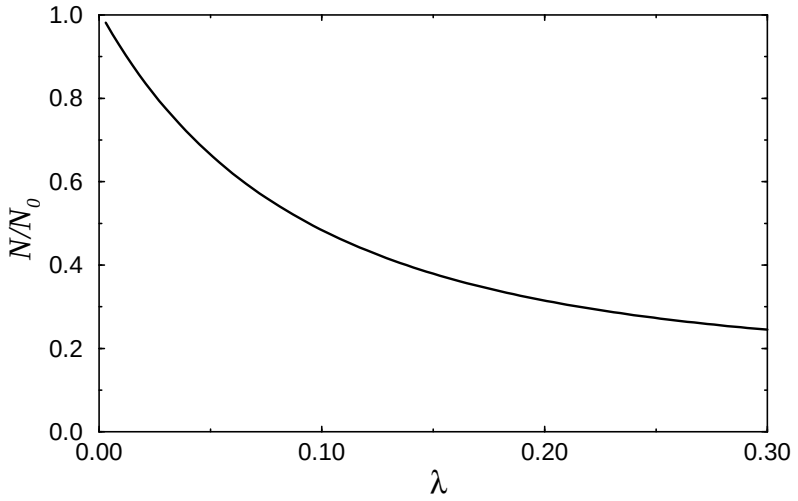

Figure 1. The number of converged levels as a function of the shape parameter λ.

Table 1. The classical value ρ_2^{cl} for different shape parameters λ. The third and the fourth columns are the best-fitting values for the Berry–Robnik and Brody model, respectively (see further text).

λ	ρ_2^{cl}	ρ_2^{BR}	β
0.000	0.00	0.18	0.00
0.010	0.05	0.26	0.02
0.017	0.15	0.28	0.03
0.022	0.26	0.32	0.05
0.025	0.34	0.35	0.09
0.028	0.58	0.36	0.03
0.033	0.75	0.41	0.15
0.040	0.86	0.49	0.16
0.049	0.92	0.68	0.48
0.055	0.99	0.78	0.69
0.062	0.99	0.77	0.87
0.070	0.99	0.94	1.42
0.100	1.00	0.97	1.58
0.200	1.00	0.99	1.93

Using the Weyl formula the levels have been unfolded to unit mean level spacing and nearest neighbour level spacing distribution $P(S)$ and number variance Σ^2 have been calculated. Instead of $P(S)$ the cumulative level spacing distribution $W(S) = \int_0^S P(s)\,\mathrm{d}s$ is shown in figure 2. In the insets we show the so-called U-functions [10, 9] which are explained in more detail below. By fitting data *locally* ($S \leqslant 0.5$) to the Brody distribution, $P^\beta(S) = aS^\beta \exp(-bS^{\beta+1})$ [21], the behaviour of $P(S)$ for small S in the transition region $0 \leqslant \lambda \leqslant 0.2$ has been analysed. However, we know that the Brody distribution cannot *globally* (for all S) capture the physical $P(S)$ (at least for $\beta > 1$). Nevertheless, for small spacings, $S < 0.5$, we found good agreement with the fractional power-law level repulsion which is provided by the Brody distribution. The results are shown in figure 3 and in table 1 and figure 5. In the figure 3 there is the T function shown, which transforms the Brody distribution to the straight line: $T(\ln S) = \ln[-\ln\{1 - W(\exp(\ln S))\}]$. Fitting the numerical data to the line,

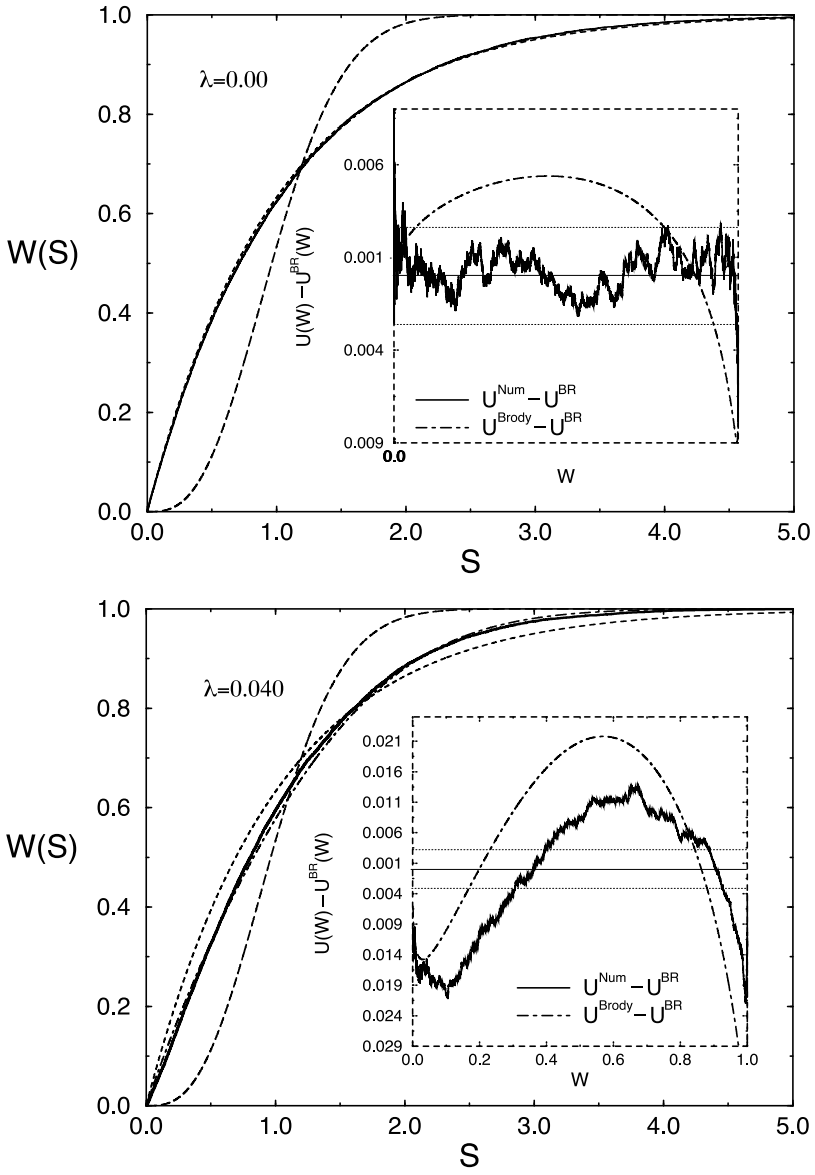

Figure 2. The cumulative level spacing distribution $W(S)$. Numerical data: full curve, Poisson: short-broken curve, GUE: long-broken curve and Berry–Robnik: chain curve. For the definition of the U functions plotted in the insets, see text.

the slope is the exponent of the level repulsion plus one. The statistical error of the measured values is shown with the bars. We also note an interesting observation, namely at a certain point in the transition region between Poissonian to GUE statistics, namely for $\lambda \approx 0.06$ (where classical dynamics is already almost fully chaotic!), the intermediate level spacing statistics is very close to the GOE statistics or the Brody distribution with $\beta \approx 1$. Please see figure 2, for $\lambda = 0.055$. However, this is the only region where the Brody distribution, except for the limiting case $\beta = 0$, has certain global relevance for non-small spacings S. A good agreement

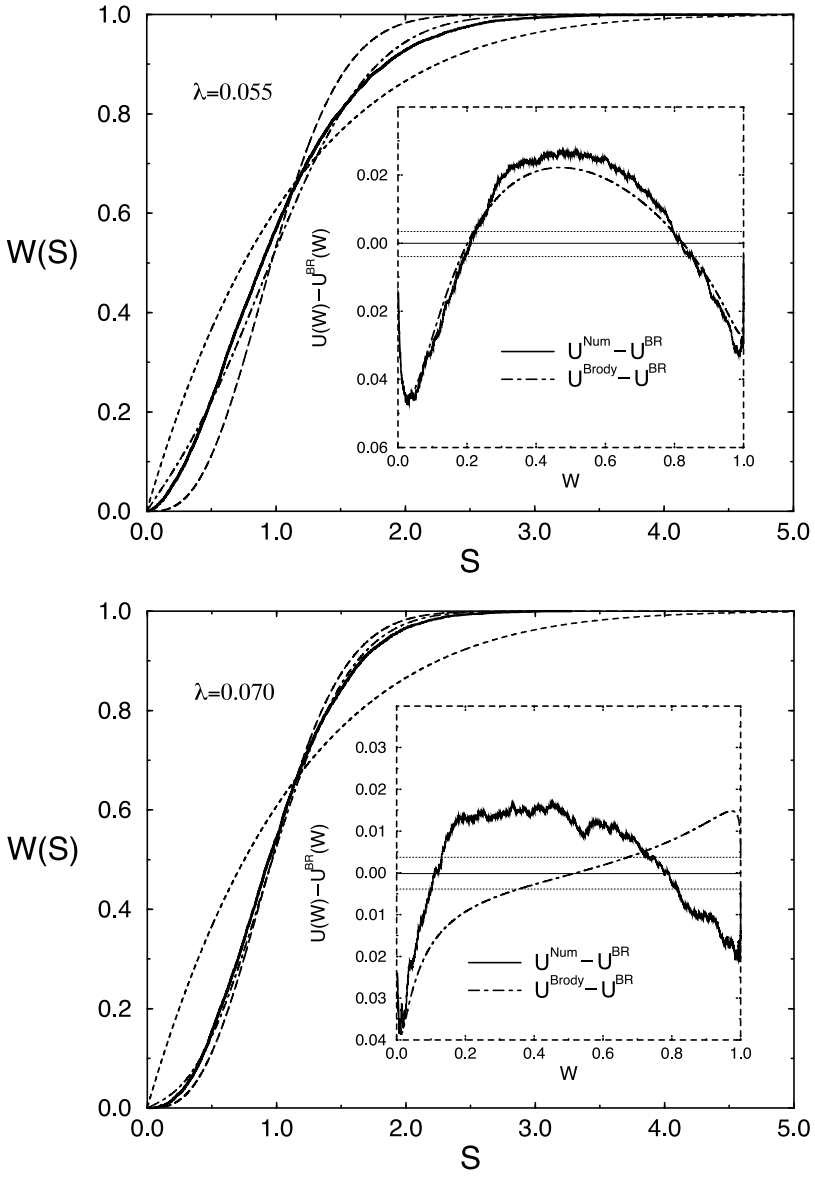

Figure 2. (Continued)

with the Poisson statistics is seen in the integrable case ($\lambda = 0$), while small deviations from GUE statistics in the fully chaotic case are observed. The deviations for $0.055 < \lambda < 0.2$ are probably due to localization of the wavefunctions, e.g. scars, and further work is necessary to explain why the agreement with GUE statistics for $\lambda = 0.2$ is highly statistically significant. For small S the quadratic level repulsion is verified at $\lambda = 0.2$. The fractional power-law level repulsion with the exponent β varying continuously from 0 to 2 is observed. From figure 3 we can also confirm that the Brody distribution is only valid locally, for $S < 1$.

As for the number variance $\Sigma^2(L)$, we have observed a continuous transition from Poisson, $\Sigma^2_{Poisson}(L) = L$, to GUE, $\Sigma^2_{GUE}(L) \approx 0.10 \ln L + 0.34$, statistics (figure 4). The saturation

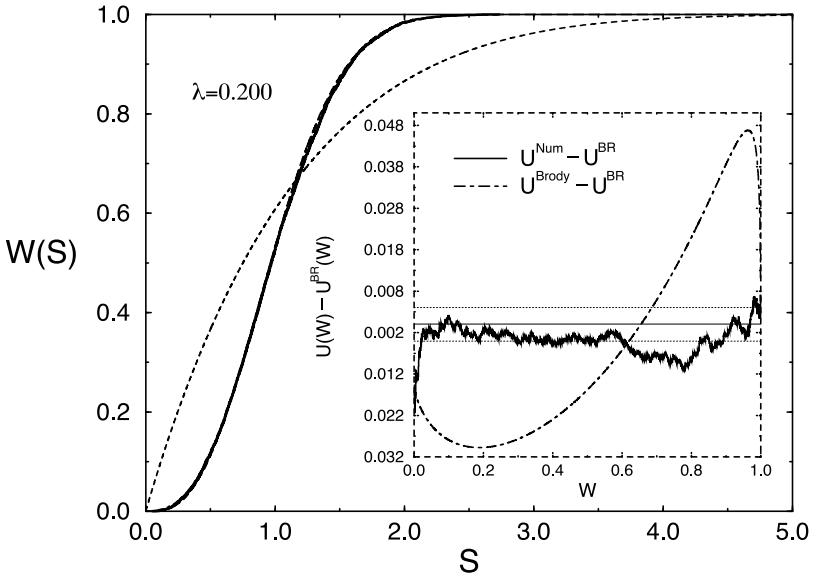

Figure 2. (Continued)

sets in at $L^* \sim 20$, which is roughly in agreement with Berry's semiclassical theory [22]. In the fully and strongly chaotic case $\lambda = 0.2$ we have found very nice agreement with GUE, however, for $L > L^*$, we observe a small but significant disagreement of numerical data with GUE and non-universal oscillations of $\Sigma^2(L)$.

We also explored the relevance of the Berry–Robnik theory, which is based on the statistically independent superposition of Poissonian sequence of regular levels and GUE sequence of chaotic levels, each having statistical weight equal to the relative measure of the corresponding classical invariant component.

Following the procedure in [15] we derived the two-component Berry–Robnik formula for Poisson–GUE transition:

$$P^{\mathrm{BR}}(S) = \mathrm{e}^{-\rho_1 S}\left[\left(\frac{32}{\pi^2}\rho_2^4 S^2 + \frac{8}{\pi}\rho_1\rho_2^2 S + \rho_1^2\right)\mathrm{e}^{-\frac{4}{\pi}\rho_2^2 S^2} + (2\rho_1\rho_2 - \rho_1^2\rho_2 S)\mathrm{erfc}\left(\frac{2}{\sqrt{\pi}}\rho_2 S\right)\right] \tag{6}$$

where $\rho_1 = 1 - \rho_2$ is the relative measure of the regular part of the classical phase space. The classical work is to determine ρ_1^{cl} and ρ_2^{cl}. Different methods and difficulties connected to this work are presented in [23]. After careful study the following method has been chosen. The surface of section† is first divided into a mesh of $M \times M$ cells. A trajectory is then run on the largest chaotic component. Each cell i has a counter which counts the number of times a trajectory passes the cell, η_i. After having made enough steps the distribution of the numbers η is plotted. The distribution has a peak around $\eta = 0$, a minimum and another maximum for larger values of η. If the distribution is normalized, the area to the right of the minimum is equal to ρ_2. In table 1 there are values of ρ_2 for different shape parameters λ.

The quantum parameter ρ_2^{BR} is determined through fitting to equation (6) and should be compared to the classical measure ρ_2^{cl}. The results are presented in figure 5 and in table 1. In the insets of figure 2 the fine scale deviations from the Berry–Robnik formula

† Surface of section, chosen in the standard way, see [12, 17].

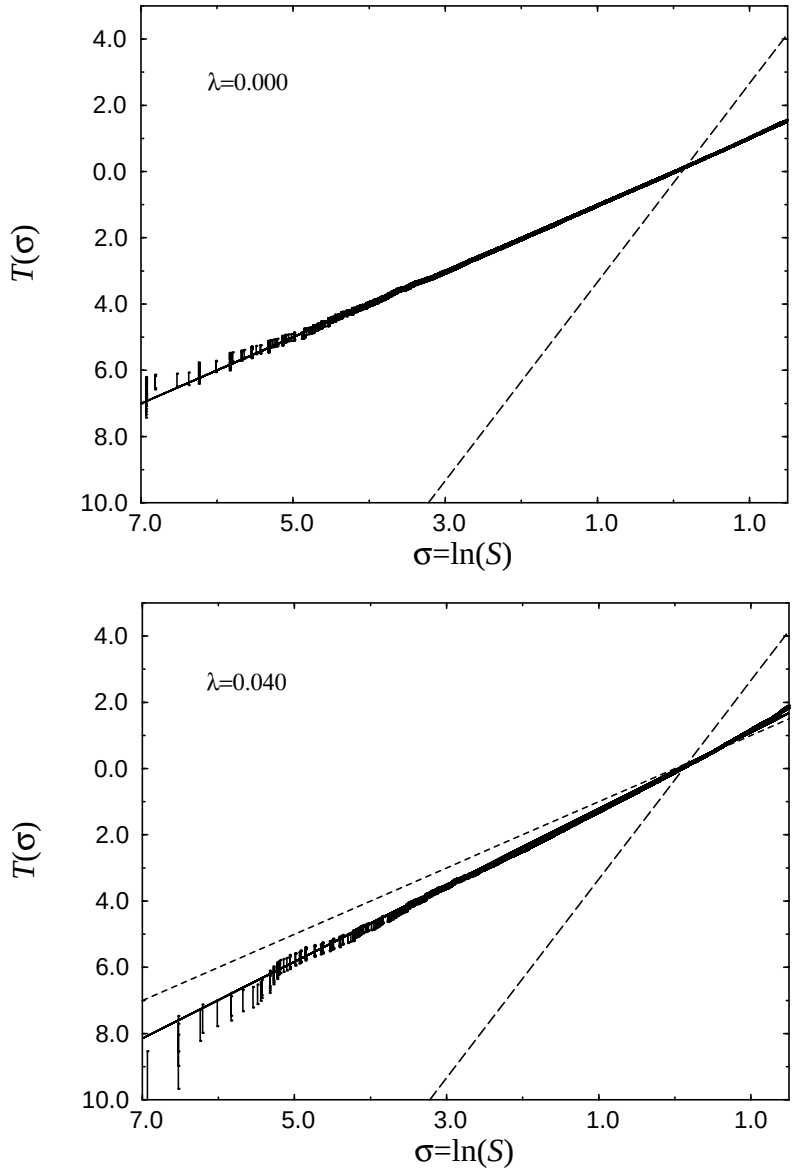

Figure 3. The function $T(\ln S)$ for the some of the calculated spectra as in figure 3. Short-broken lines give the Poissonian distribution, while the long-broken lines are the limiting Brody distribution (for $\beta = 2$) which is, however, incorrect in the tail.

are plotted in terms of the $U(W(S)) - U(W^{BR}(S))$ versus $W(S)$ (full curve). We also plot $U(W^{Brody}(S)) - U(W^{BR}(S))$ for comparison (broken curve). The transformation $U(W) = \frac{2}{\pi}\arccos\sqrt{1-W}$ is used [9] in order to have uniform statistical error over the plot, $\delta U = 1/(\pi\sqrt{N})$. The dotted horizontal lines indicate $\pm\delta U$. The Berry–Robnik fit is not statistically significant and the values of the fitting parameters ρ_2^{BR} are different from the classical values ρ_2^{cl}. Why this is so and how the crossover from Brody-like behaviour to Berry–Robnik takes place is discussed in [8]. As one can see, at small S, (and therefore at

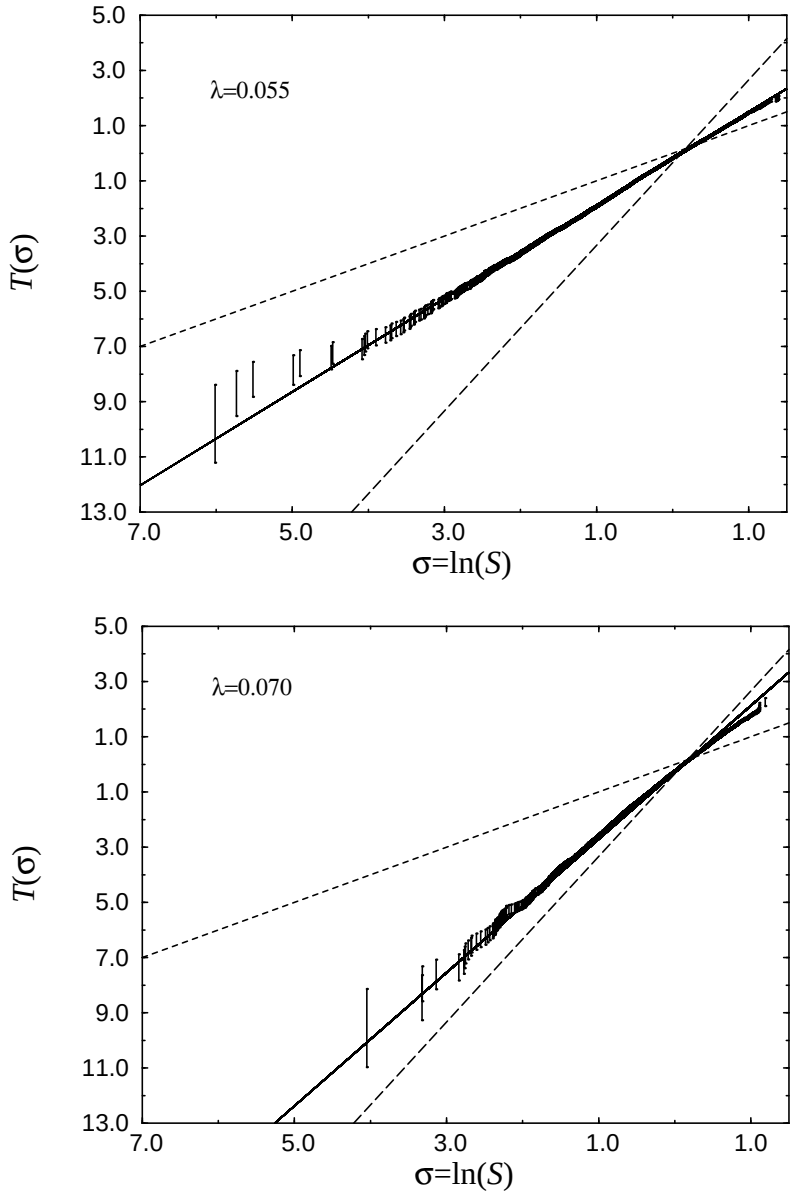

Figure 3. (Continued)

small $W(S)$) the agreement with Brody is better than Berry–Robnik, and occasionally, like e.g. at $\lambda = 0.055$ and 0.040, even globally Brody seems to capture the behaviour of our data, although, on the other hand, there are *expected deviations* at large S (and large $W(S)$) e.g. in the fully chaotic case of an Africa billiard ($\lambda = 0.2$), where $\beta = 1.93$, and Brody asymptotic behaviour at large S is certainly wrong, because its exponent varies $\propto S^3$, which is different from GUE in equation (4), $\propto S^2$. What we see in this U-function representation is consistent with the T-function representation in figure 3.

In this paper we have presented the numerical computation and statistical analysis of

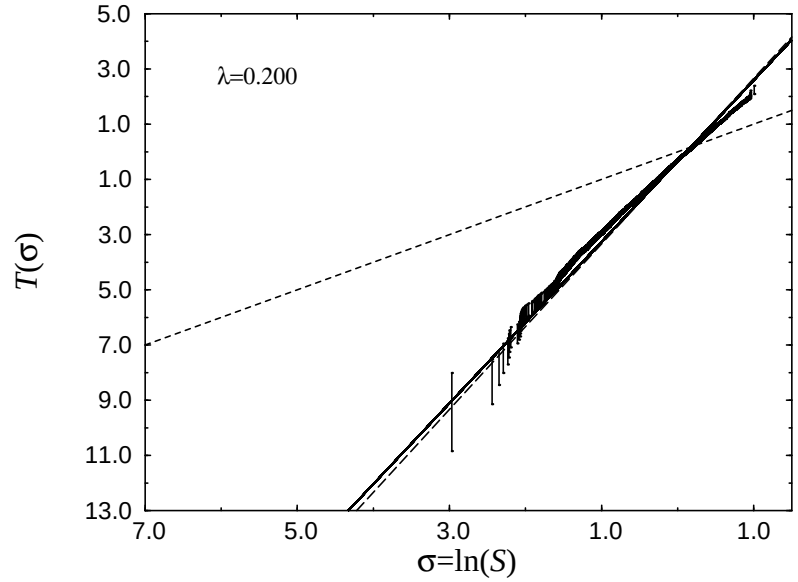

Figure 3. (Continued)

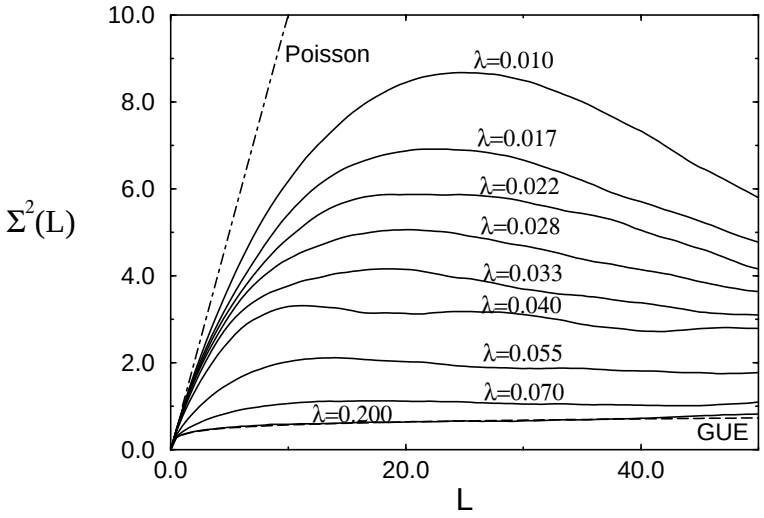

Figure 4. The number variance statistics $\Sigma^2(L)$ for different shape parameters. The limit cases are shown as chain (Poisson) and broken (GUE) curves.

energy spectra of a Hamiltonian system with mixed classical dynamics and without anti-unitary symmetry, namely the Aharonov–Bohm cubic conformal plane billiard. We reconfirmed the GUE statistics in the ergodic (fully chaotic) case with approximately 6000 good consecutive energy levels whose accuracy is better than one per cent of the mean level spacing. Further, in the KAM regime we have found *the fractional power-law level repulsion* with the exponent varying smoothly from zero to two as the corresponding classical dynamics goes from integrable to ergodic. This is an important phenomenon, which cannot be understood in terms

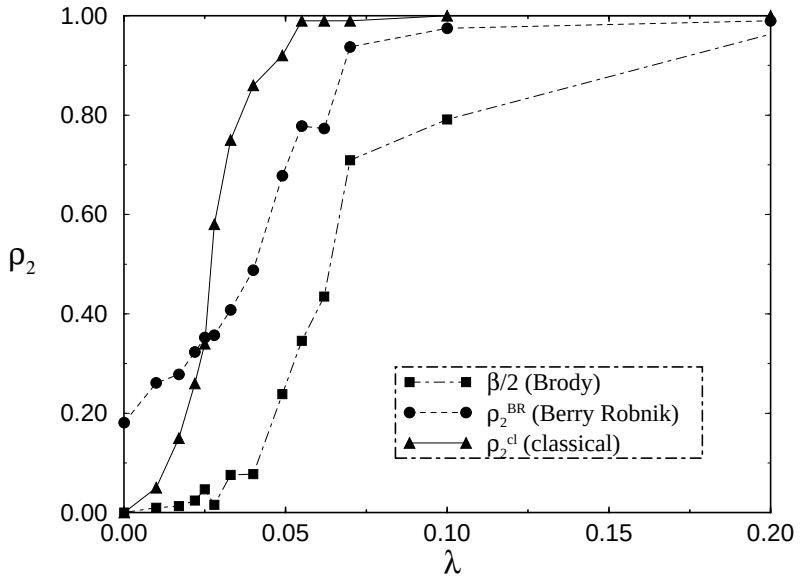

Figure 5. Calculated values of ρ_2^{cl}, ρ_2^{BR} and $\frac{1}{2}\beta_{Brody}$ as functions of λ.

of relatively simple and straightforward modelling within the framework of random matrix theories where one studies the Poisson–GUE transition predicting the quadratic level repulsion. In mixed dynamical systems at not too high energies we should observe the quadratic level repulsion only at exponentially small energy level spacing scales $S \leqslant \Delta S \propto \exp(-\text{const}/\overline{h})$, which is very difficult. At intermediate scales S, $\Delta S \leqslant S \leqslant 1$, we should observe the fractional power-law level repulsion, whose origin is in the localization phenomena of chaotic eiegenstates or/and is due to the quantal KAM structure. These dynamical phenomena are not captured by the above-mentioned random matrix theories.

On the other hand we know that in the ultimate semiclassical limit the Berry–Robnik theory [15, 8] should apply, which does not exhibit level repulsion ($P^{BR}(S=0) = \rho_1^2 \neq 0$). The resolution of this disagreement lies in the fact that the key assumptions of the Berry–Robnik theory are fulfilled only for extremely large sequential quantum numbers or small effective Planck's constant. Namely, one should require extendedness of the Wigner functions on the whole corresponding classically allowed invariant components of phase space. This is only true if the *Heisenberg time*, $t_H = 2\pi\overline{h}/\langle\Delta E\rangle$ (the time until the discreteness of the spectrum is not yet resolved and the quantum evolution follows the classical one), is larger than the classical *ergodic time*, t_e (the typical time after which classical distributions reach equilibrium steady state on the invariant set). In our case we have typically $t_e \gtrsim 10^4$ (velocity is one), so the condition $t_H > t_e$ yields that the sequential number N should be larger than about 10^8. We believe that this paper contributes to the new developements in the wide field of research of quantum chaos, recently reviewed by Weidenmüller *et al* [19]. Specifically, some of the main issues concerning the Poisson–GOE transiton in dynamical systems have been recently addressed by Robnik and Prosen [8] and reviewed by Robnik [25].

Acknowledgments

JD would like to thank Darko Veberič for reading the manuscript. This work was supported by the Ministry of Science and Technology of the Republic of Slovenia and by the Rector's Fund of the University of Maribor.

References

[1] Tomsovic S 1986 *PhD Thesis* University of Rochester
[2] French J B, Kota V K B, Pandey A and Tomsovic S 1988 *Ann. Phys., NY* **181** 198
[3] Leyvraz F and Seligman T H 1990 *J. Phys. A: Math. Gen.* **23** 1555
[4] Guhr T and Müller-Groeling A 1997 *J. Math. Phys.* **38** 1870
[5] Robnik M 1987 *J. Phys. A: Math. Gen.* **20** L495
[6] Izrailev F 1993 Private communication
[7] Prosen T 1993 unpublished
[8] Robnik M and Prosen T 1997 *J. Phys. A: Math. Gen.* **30** 8787
[9] Prosen T and Robnik M 1993 *J. Phys. A: Math. Gen.* **26** 1105
[10] Prosen T and Robnik M 1994 *J. Phys. A: Math. Gen.* **27** 8059
[11] Robnik M 1986 *Lecture Notes in Physics* **226** 120
[12] Berry M V and Robnik M 1986 *J. Phys. A: Math. Gen.* **19** 649
[13] Robnik M 1992 *J. Phys. A: Math. Gen.* **26** 1399
Robnik M 1992 *J. Phys. A: Math. Gen.* **26** 3593
[14] Bohigas O, Giannoni M J and Schmit C 1984 *Phys. Rev. Lett.* **226** 120
[15] Berry M V and Robnik M 1984 *J. Phys. A: Math. Gen.* **17** 2413
[16] Bohigas O, Tomsovic S and Ullmo D 1993 *Phys. Rep.* **223** 43–133
[17] Dobnikar J 1996 Spectral statistics in the transition region between integrability and chaos for systems with broken anti-unitary symmetry *Diploma Thesis* CAMTP and Physics department, FMF, University of Ljubljana, unpublished
[18] Robnik M 1997 *Open Syst. Inform. Dynam.* **4** 211
[19] Guhr T, Müller-Groeling A and Weidenmüller H A 1998 *Phys. Rep.* **299** 189
(Guhr T, Müller-Groeling A and Weidenmüller H A 1997 Random matrix theories in quantum physics: common concepts *MPI Preprint* H V27 MPIfK Heidelberg, cond-mat/9707301)
[20] Robnik M 1984 *J. Phys. A: Math. Gen.* **17** 1049
[21] Brody T A 1973 *Lett. Nuovo Cimento* **7** 482
[22] Berry M V 1985 *Proc. R. Soc.* A **400** 229
[23] Robnik M, Dobnikar J, Rapisarda A, Prosen T and Petkovšek M 1997 *J. Phys. A: Math. Gen.* **30** L803
[24] Haake F 1992 *Quantum Signatures of Chaos* (Berlin: Springer)
[25] Robnik M 1998 *Nonlinear Phenomena in Complex Systems (Minsk)* **1** 1